

Descending Price Algorithm for Determining Market Clearing Prices in Matching Markets

Shih-Tang Su, Vijay Subramanian
University of Michigan

Abstract—In this paper, a two-sided matching market is considered, and a descending price algorithm based auction mechanism is proposed to determine market clearing prices. For general valuations with a intelligent choice of reverse constricted sets, we prove that the algorithm converges in a finite number of rounds. Then specializing to the rank one valuations in sponsored search markets, we provide an alternate characterization of the set of all market clearing prices, and use this to prove that the proposed algorithm yields the element-wise maximum market clearing price. We then conclude with a discussion of some special cases and counter-examples in order to get a better understanding of the algorithm.

Keywords—Matching market, sponsored search market, descending price auction, VCG mechanism

I. INTRODUCTION

The problem of matching in markets was first studied by Gale and Shapley in the seminal paper "College Admissions and the Stability of Marriage" [1]. Soon thereafter a widely used mathematical model of two-sided matching markets was presented in the paper "The Assignment Game I: The Core" [2] by Shapley and Shubik. The setup consists of a set of sellers and a set of buyers. Each seller comes with one indivisible good and seeks to trade with an unit-demand buyer to earn the most money. During the ensuing decades, the matching market model has been widely studied, and applied to many areas such as commodities trading, matching students to schools, and organization of organ donors. A comprehensive account of the work on this topic can be found in the following surveys [3], [4], [5].

In the assignment game model of matching markets of [2], the authors identify a set of prices that lead to a perfect matching between the sellers and the buyers, i.e., assignments such that every seller is allocated to a buyer, and every buyer obtains a good; any price vector in this set is called a market clearing price. The set of market clearing prices is further shown to be a solution of a linear programming problem (dual to the optimal assignment problem), and also shown to constitute the core of the assignment game; this characterization shows that any market clearing price will yield the optimal assignment, i.e., social-welfare maximizing assignment (see Chap. 15 of [6] and Chap. 5 of [7]). Finally, the set of market clearing prices is also shown to have a lattice structure. Given this characterization of market clearing prices, many algorithms have been developed to find them; and an auction-based method to find market clearing prices is generally a preferred solution. An ascending price based auction approach was presented by Demange, Gale,

and Sotomayor in [8], which concluded the line of work in Demange's auction design relying on Hungarian algorithm [9], [10]. The ascending price auction approach also corresponds to a primal-dual algorithm for the linear programming problem [10], Chap. 5 in [7]. Using sub-gradient algorithms for linear programming, there exists an alternate ascending price auction procedure to determine market clearing prices (see Chap. 5 in [7]). It also shown (Chap 15 of [6], Chap. 5 of [7], and [11]) that the personalized prices produced by the Vickrey-Clarke-Groves mechanism [12], [13], [14] are market clearing posted prices when the price attributed to each buyer is associated with the price of seller she is matched to. Furthermore, the both ascending price algorithms in Chap. 5 of [7], [8] can also be modified in such a way that the VCG prices are obtained; tying the ascending price algorithms to standard algorithms for linear programming problems also shows that the VCG prices can be obtained with reasonable computational complexity. We note here that to make the step-size choice of the algorithms be easy, it is often assumed that the valuations are all integers. It is also easily shown [11] that the set of market clearing prices satisfies a lower lattice property, i.e., component-wise minimum of two market clearing prices is also a market clearing price. This then immediately implies that there is a minimum element in the set of market clearing prices. It has been shown [10] that the VCG prices yield the smallest sum of prices among all market clearing prices, and hence, correspond to the minimum element.

When specialized¹ to a single good/seller and many buyers, the ascending price algorithm corresponds to an English auction and is equivalent to the second-price auction (or Vickrey auction [12]) in Chap. 15 of [6] and Chap. 5 of [7]. For the single good setting, we also have a Dutch auction, which is a descending price algorithm and corresponds to a first-price auction (see Chap. 9 in [6]). It is obvious that the price produced by the descending price algorithm is the maximum possible market clearing price. Connecting back to the original assignment game work [2], we already know that the set of market clearing prices is a compact convex set. Therefore, it has a Pareto boundary (see Chap. 11 of [6] and Chap. 2 of [7]). Furthermore, following the work of [2], it is also known that an upper lattice property also holds for set of market clearing prices (closure holds when taking

¹This can be made into a matching market by adding dummy sellers for whom all buyers have valuation 0.

the element-wise maximum of two valid price vectors), so that a unique maximum element exists in the set of market clear prices and the Pareto boundary is a singleton element. Maximizing linear functions using versions of convex programming algorithms, it is then possible to determine the Pareto boundary. However, the Dutch auction procedure strongly motivates us to find a suite of auction procedures that will yield the maximum market clear price vector. To further the connection with the Dutch auction, we are again motivated to consider descending price algorithms.

The VCG prices are dominant strategy and incentive compatible, so that agents will truthfully reveal their valuations. On the contrary, agents shade the bids in a first-price auction (in a Nash implementation) (see Chap. 11 of [6] and Chap. 4 of [7]). For the current paper, we will not consider these strategic concerns and take the valuations as known. The analysis of the strategic behavior under descending price algorithms is for future work.

In the family of matching markets, there is a subset named "sponsored search markets," which is an extremely productive research topic given its relevance to search engine advertisement displays [15]. Specifically, the sponsored search market applies to the setting wherein the search engine sells web-display advertisement slots to advertisers. An instance of the sponsored search market is generated every time one types a keyword and clicks the search button. The search engine not only provides the related websites to the query but also displays a list of advertising links, and the market making takes place to sell these links.

The specific structure of the sponsored search market makes the analysis much easier: VCG prices can be obtained in closed-form; and many locally envy-free mechanisms have been proposed [16], [17] to obtain (component-wise) higher market clearing prices with different properties when agents are strategic. We will also analyze the sponsored market setting to determine any simplifications that arise for descending price algorithms. By taking some realistic issues into consideration, scenarios with reserve prices [18] and analysis of strategic concerns via BayesNash equilibria [19] [20] have been widely discussed. We will, however, defer such considerations for future work.

A. Our contribution and Outline

In this paper, we propose a descending price algorithm in search of market clearing prices in matching markets. Then proving the proposed algorithm will terminate in a finite number of rounds (linear in the number of agents). We also investigate the properties of final prices obtained from our algorithm when specialized to sponsored search markets. We also find an alternate characterization of the set of market clearing prices in sponsored search markets, and using this prove the proposed descending price algorithm will return the (element-wise) maximum market clearing price; in other words, prove that the Pareto boundary is a single element, and develop an auction to achieve it.

The outline of the paper is as follows. In Section II we set up the notation and mathematically describe the problem

formulation. We develop the descending price algorithm in Section III and also prove convergence properties of the algorithm. We then specialize the results to sponsored search markets in Section IV. Section V, Section VI, Section VII contain some discussions of the algorithm, brief description of future work, and conclusions, respectively.

II. PROBLEM FORMULATION

We consider a matching market with a set M of m distinct goods with exactly one copy of each good available for sale. We also assume that there is a set of B buyers with each buyer $i \in B$ having a non-negative valuation $v_{ij} \geq 0$ for good $j \in M$, and desiring at most one good among all the goods available. We can collect all the valuations together in a $|B| \times |M|$ valuation matrix V . Since we can always add dummy goods (zero private value for every buyer) or dummy buyers (zero value on every good), we can assume $|B| = |M| = m$ without loss of generality; henceforth, this will be in force.

In general, the valuations of the buyers for the goods are their private values, and in an auction these are revealed via bids made by the buyers, but the bids may not be truthful. In this paper we will ignore strategic concerns in deciding the bids, and will assume that the buyers reveal their valuations truthfully. The mechanism presented here and the analysis of it will be used in future work to determine exactly how the buyers use their private valuations and knowledge of the mechanism to determine their bids.

We will assume quasi-linear utilities for the buyers, and therefore assume that prices can be used for market making. Whereas most of our discussions will be for posted prices, personalized prices obtained via a mechanism such as the VCG mechanism can also be interpreted as a posted price by assigning the per buyer transfer (taxation) to the good that the buyer is matched to. Given a price vector $P = [P_1 P_2 \dots P_m]$, we define $U_{i,j} = v_{i,j} - P_j$ as the payoff of buyer i if good j is allocated to it. Since each buyer has unit demand, we define U_i^* be the maximum payoff of buyer $i \in B$, i.e. $U_i^* = \max\{0, \max_{j \in M} v_{i,j} - P_j\}$. For a buyer i we define its set of preferred goods P_i to be the set of goods that yield payoff U_i^* if assigned to buyer i . Note that the preferred goods set of a buyer can be empty, and this occurs if its payoff for all the goods is negative. Then, by connecting each buyer with its preferred goods, we can construct a bipartite graph.

Definition 1. A *bipartite graph* is a graph such that all vertices can be divided into two disjoint subsets, and there are no edges connecting vertices in the same subset.

After constructing the bipartite graph based on a specific price vector, we can allocate each buyer one of goods it prefers. If every good can be allocated and each buyer obtains a good, we say that this bipartite graph has a perfect matching.

Definition 2. A bipartite graph is said to possess a *Perfect Matching*, if there is a sub-graph such that each node is connected by an edge it is assigned to, and no two nodes on one side are assigned to the same node on the other side.

Note that the bipartite graph we construct depends on both the valuation matrix and the price vector. Given a specific valuation matrix, we called a price vector that leads to a bipartite graph with a perfect matching a market clearing price vector. The main goal of this paper is to understand the set of market clearing prices, and in particular determine the Pareto optimal boundary of this set. We do this by presenting an algorithm that searches for market clearing prices in matching markets using a descending price auction. In other words, we present a procedure that generalizes a Dutch auction for a single item.

Prior to introducing our algorithm, we restate some definitions and review some existing results related to our work in matching markets.

Definition 3. *The neighbor of a set, denoted by $N(S)$, is the largest set that for every vertex $i \in N(S)$, there is an edge connecting i to at least one vertex in the set S .*

When constructing bipartite graphs of matching markets with goods and buyers, it is common to put goods on the left-hand side, and the buyers on the right-hand side. Next, we review the definition of a constricted set, and also define a reverse constricted set below. For a set S of either buyers or sellers but not both, define by $N(S)$ all the agents on the other side that are connected to agents in S .

Definition 4. *A right-hand side set S in a bipartite graph is called a **constricted set** if its size is greater than the size of its left-hand neighbors, i.e. $|S| > |N(S)|$.*

Definition 5. *A left-hand set S in a bipartite graph is called a **reverse constricted set** if its size is greater than the size of its right-hand neighbors.*

Let's recall Hall's marriage theorem [21].

Hall's marriage theorem: A bipartite graph G with vertex sets V, U and an edge set E contains a perfect matching from V to U if and only if it satisfies Hall's marriage condition

$$|N(S)| \geq |S| \text{ for every } S \subseteq V,$$

where $N(S) \subseteq U$ is the set of neighbors of S in G .

According to Hall's marriage theorem, if a bipartite graph has either a constricted set or a reverse constricted set, it will not have a perfect matching.

Given any valuation V , it is well known that the set of market clearing prices is non-empty and bounded [2]. Boundedness is obvious from the finiteness of the valuations. Non-emptiness is established either using the characterization in [2] or using a constructive ascending price algorithm [8] that starts from all the prices being 0, and also by using the prices determined by the VCG mechanism (see Chap 15 in [6]) for this problem.

Before reviewing the existing ascending price algorithm, we recall the definition of preferred-seller list, and define the lined-up-buyer list from the seller's perspective in the same bipartite graph.

Definition 6. *Under a set of price \mathbf{P} , the **preferred-seller list** of buyer i is a set of goods S such that $v_{i,j} - P_j = U_i^*$*

for all $j \in S$.

Definition 7. *Under a set of price \mathbf{P} , the **lined-up-buyer list** of good j is a set of buyers S such that $v_{i,j} - P_j = U_i^*$ for all $i \in S$.*

In [6], the author considered a integer based valuation matrix, and proposed an algorithm to find market clearing prices by eliminating constricted sets of buyer. In the ascending price auction, the algorithm finds a constricted set of buyers and increases the price of every good in the neighbor of constricted set by one. This algorithm can find the market clearing prices for any integer-based valuation matrix. Similarly, the authors in [6] also show that the VCG mechanism when applied to the matching market also produces market-clearing prices (by assigning the user prices to the goods that they're matched to) with the sum total price being the lowest of all market clearing prices. Both of these points generalize results for the auction of a single good where the ascending price auction corresponds to an English auction. As mentioned earlier our goal to find the corresponding generalization of the first-price auction and the Dutch auction.

III. DESCENDING PRICE ALGORITHM FOR GENERAL VALUATION MATRIX

In this section, we will propose a suite of descending price algorithms in search of market clearing prices and present an analysis of a specific version in the suite of algorithms proposed. We will start by choosing the prices to be an easily established upperbound price vector. In contrast to the ascending price algorithm we will be working with reverse constricted sets of goods and will be reducing their prices appropriately. We will then pick a specific algorithm to analyze by specializing the choice of the reverse constricted set to a specific choice, and also explain the criterion used. We will then prove the finite-time convergence of the proposed algorithm, which will then prove that we obtain market clearing prices.

A. Exploration of Descending Price Algorithms

According to Hall's marriage theorem, if a bipartite graph does not have a perfect matching, we can always find reverse constricted sets. Hence, we can mimic the technique used in Chap. 10 of [6] to find market clearing prices to give a framework to produce a suite of descending price algorithms by reducing the price of goods in a reverse constricted set, and hence, eliminating them at each step of the algorithm.

1) Initial Price Choice and Optimal Price Reduction:

Note that we have to specify each step of Algorithm 1 in order to make it a workable (and analyzable) algorithm. The first step is to set a "good" initial price vector for any descending price auction. The first condition that the initial price vector has to satisfy is that it will be (component-wise) greater than or equal to any possible market clearing price. Besides we want the initial price vector to be the lowest possible such vector in every component to prevent wasteful iteration checking some price vectors that cannot be market clearing prices.

Algorithm 1 Descending Price Algorithm Framework**Input:** A $|B| \times |M|$ valuation matrix \mathbf{V} .**Output:** Vector of Market Clearing Price \mathbf{P}

- 1: Initialization, set initial price.
- 2: Construct the lined-up-buyer list graph.
- 3: **while** There exists a reverse constricted set **do**
- 4: Pick a reverse constricted set.
- 5: Reduce the price by some amount.
- 6: Check all prices are non-negative.
- 7: Construct the lined-up-buyer list bipartite graph.
- 8: **end while**
- 9: Return \mathbf{P} .

By definition, if there exists a market clearing price vector, good j must be allocated to a buyer. Hence, $v_{i,j} - P_j \geq 0$ for some i . If we set $P_j > \max_{i \in M} v_{i,j}$, the preferred buyer set of good j will be an empty set. Thus, P_j definitely cannot be a market clearing price choice for good j . The highest possible P_j to be a market clearing price is $P_j = \max_{i \in M} v_{i,j}$. Note also that this is the lowest possible upperbound on a market clearing price for good j . Hence, we set $P_j = \max_{i \in M} v_{i,j}$, $\forall j \in M$ as the initial price vector.

Second, we want to find the optimal price reduction in each round of the iteration, again in order to avoid wasteful iterations. In order to eliminate the reverse constricted set, the price reduction should be large enough to attract at least one buyer not in the neighbor of reverse constricted set to line up to a good in the reverse constricted set. By this procedure the price reduction cannot be too large to exclude some possible market clearing prices. We determine the optimal price reduction in Lemma 1.

Lemma 1. *Given a reverse constricted set with goods S and current price vector \mathbf{P} , the minimum price reduction of all goods in this reverse constricted set that is guaranteed to add at least a new buyer to the set is*

$$\min_{i \in B \setminus N(S), l \in S} \left\{ \max_{k \in M \setminus S} (v_{i,k} - P_k) - (v_{i,l} - P_l) \right\}. \quad (1)$$

Proof: Please see Appendix A in [22]. ■

The logic of Lemma 1 is that we want to find the minimum value to compensate the buyers not in the $N(S)$ to make at least one buyer indifferent between one of the goods in $N(S)$ and the good(s) she prefers initially.

2) *Skewness criterion for choosing reverse constricted sets:* Different choice of reverse constricted sets may generate different market clearing prices, assuming that the algorithm converges in finite time. However, we want our algorithm to return the same set of market clearing prices given the same input valuation matrix. In order to get a unique set of market-clearing prices, a straightforward idea is to restrict the choice of reverse constricted sets, and pick a specific one at any given iteration of the algorithm. On one hand, as in the example in Figure 1 we prefer choosing the reverse constricted set in UL rather over UR because we want to choose the largest constricted set given the same set of neighbors (buyers). On the other hand, we prefer DL

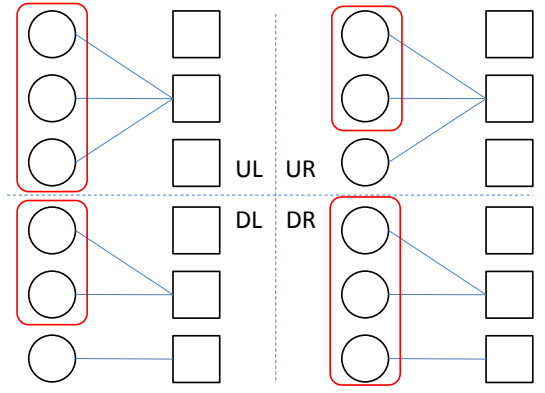


Fig. 1: Criteria for choosing reverse constricted sets

rather than DR because we do not want to increase the size of reverse constricted set by just merging a reverse constricted set with a non-constricted one. Additionally, unlike in the ascending price algorithm setting we don't have a proof of convergence of any algorithm from the suite of descending price algorithms, but restricting the choice of the reverse constricted set allows us to prove convergence in finite time.

With this intuition in mind, we add two rules for choosing constricted sets:

- 1) Pick the reverse constricted sets with goods S with the largest difference $|S| - |N(S)|$.
- 2) If there are multiple reverse constricted sets with the same $|S| - |N(S)|$, then choose the reverse constricted set with the smallest size.

These two rules are equivalent to searching for the most "skewed" reverse constricted set in the bipartite graph by defining a function to represent the skewness of a set.

Definition 8. *The skewness of a set of goods S in a matching market is defined by function $f : 2^M \setminus \emptyset \mapsto \mathbb{R}$ with $f(S) = |S| - |N(S)| + \frac{1}{|S|}$ for all $S \subseteq M$ with $S \neq \emptyset$, where 2^M is the power set of M .*

Lemma 2 below then shows that the most skewed set and the most skewed reverse constricted set are the same, if the bipartite graph does not have a perfect matching.

Lemma 2. *If a reverse constricted set exists in a bipartite graph, then the most skewed set is a reverse constricted set.*

Proof: Please see Appendix B in [22]. ■

While the property in Lemma 2 is valuable, it still does not ensure that adding the new rules makes sure that our algorithm always returns the same market clearing prices, given the same valuation matrix. Therefore, we want to show that in each round of our algorithm, the most skewed set is unique. Lemma 3 proves the uniqueness of the most skewed set given an arbitrary bipartite graph for which reverse constricted sets exist, which contains the condition we need.

Lemma 3. *Given a bipartite graph with reverse constricted sets, the most skewed set is unique.*

Proof: Please see Appendix C in [22]. ■

Algorithm 2 Skewed-set Aided Descending Price Auction

Input: A $|B| \times |M|$ valuation matrix \mathbf{V} .

Output: Vector of Market Clearing Price \mathbf{P}

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1: Initialization, set the price of good  $j$ ,  $P_j = \max_{i \in B} v_{i,j}$ .
2: Construct the lined-up-buyer list graph.
3: while There exists a reverse constricted set do
4:   Choose the most skewed reverse constricted set  $\mathbf{S}$ .
5:   For all  $j \in \mathbf{S}$ , reduce  $P_j$  by
      $\min_{i \in B \setminus N(\mathbf{S}), l \in \mathbf{S}} \{ \max_{k \in M \setminus \mathbf{S}} (v_{i,k} - P_k) - (v_{i,l} - P_l) \}$ 
6:   if  $\min_{j \in \mathbf{S}} P_j < 0$  then
7:     Let  $c = -\min_{j \in \mathbf{S}} P_j$ .
8:     Update  $P_k = P_k + c$  for all  $k \in M$ .
9:   end if
10:  Construct the lined-up-buyer list graph.
11: end while
12: Return  $\mathbf{P}$ .
```

With Lemmas 2 and 3 in place, it easily follows that the two rules we imposed are equivalent to finding the most skewed set at every iteration. With the proper initial price vector choice, optimal price reduction per round, and the unique choice of the most skewed set, the complete algorithm is displayed in Algorithm 2.

B. Convergence of the Algorithm

If Algorithm 2 terminates, then by construction we will necessarily obtain a market clearing price. In this section, we will prove that the algorithm terminates in a finite number of rounds. In the algorithm, we choose the most skewed set in every round. Since there is no reverse constricted set when a perfect matching exists, the most skewed set has skewness smaller than or equal to one when a market clearing price exists; by the Hall marriage theorem, this is also a sufficient condition. Given a bipartite graph G in a matching market, we can define the skewness of the graph $W(G)$ to equal the skewness of the most skewed set. Therefore, by defining a sequence $W(G_k) = \max_{S \in M, S \neq \emptyset} f_k(S)$, where G_k is the bipartite graph obtained at the k^{th} iteration of Algorithm 2, we can show the convergence of the algorithm in a finite number of rounds by proving the sequence $W(G_k)$ is a strictly decreasing sequence and the minimum difference between adjacent two numbers in the sequence is greater than some positive constant. Note that $W(\cdot)$ is a potential function that will be shown to strictly decrease in every iteration of the algorithm. It is the basic idea of the proof of Theorem 1.

Theorem 1. *The descending price algorithm terminates in finite rounds.*

Proof: Please see Appendix D in [22]. ■

Before we proceed further with our analysis, we discuss the computational complexity of the algorithm. Given a bipartite graph, finding a most skewed set would be of exponential complexity in the number of vertices of the graph. However, it is possible that the task is simpler if there is no perfect matching. Furthermore, it is also possible that in the execution of our algorithm, this task is even simpler.

With a restriction on the valuation matrix, in the next section we will show that we can determine the most skewed set in polynomial time during the execution of the algorithm.

IV. ALGORITHM IN SPONSORED SEARCH MARKETS

In this section, we investigate a special kind of matching market called sponsored search market. In a sponsored search market, a search engine has a multiple web slots for displaying ads and the advertisers want to bid for the slots. Given the dominance of per-click pricing in Internet advertising over the traditional per-impression model [23], we assume that each slot has a specific click-through rate associated with it, i.e. c_k for slot k , and assume the advertisers' private value on slots are proportional to the click-through rate. That is, for any pair of advertiser i, j ,

$$\frac{v_{i,k}}{v_{i,l}} = \frac{v_{j,k}}{v_{j,l}} = \frac{c_k}{c_l}.$$

Therefore, the entries in the valuation matrix $v_{i,j}$ can be represented by an advertiser-dependent parameter w_i and a slot-dependent parameter c_j such that $v_{i,j} = w_i c_j$; note that the valuation matrix is now a rank one matrix. We call the advertiser-dependent parameter w_i the weight of advertiser i . Without loss of generality, we can rearrange the valuation matrix to make $c_i \geq c_j$ and $w_i \geq w_j$ for all $i < j$.

A. Set of all Market Clearing Prices

Assume $|M| = |B| = m$ and a given general valuation matrix V , determining the set of market clearing prices is a non-trivial task that needs a careful analysis of the dual of the optimal assignment problem [2]. This makes the set of market clearing prices difficult to compute and represent. However, in a sponsored search market, assigning web slot i to buyer i for all $i \in M$ is always a social welfare maximizing matching; it is also the unique such matching if the weights and click-through rates are distinct. With this property, for all $|i - j| > 1$, the constraint on $P_i - P_j$ yields a tighter bound² than the summation of such bounds from i to j . Therefore, the number of constraints of all sets of market clearing price can be reduced from m^2 to $2m$. After that we can derive the set of market clearing prices and represent it in a succinct manner.

Lemma 4. *In a sponsored search market with click-through rates $c_1 \geq c_2 \geq \dots \geq c_m$ and private weight of each advertiser $w_1 \geq w_2 \geq \dots \geq w_m$, the price vector $\mathbf{P} = [P_1 P_2 \dots P_m]$ is market clearing if and only if it belong to the set $\mathbf{Q} = \{P_i : w_{i+1}(c_i - c_{i+1}) \leq P_i - P_{i+1} \leq w_i(c_i - c_{i+1}), i \in \{1, 2, \dots, m-1\}; 0 \leq P_m \leq w_m c_m\}$, in short $\mathbf{P} \in \mathbf{Q}$.*

Proof: Please see Appendix E in [22]. ■

With the set of market clearing prices in sponsored search market determined, we have the following Corollary 1, 2.

Corollary 1. *Given the same assumption that $c_1 \geq c_2 \geq \dots \geq c_m$ and $w_1 \geq w_2 \geq \dots \geq w_m$, there is a maximum market clearing price. At this market clearing price, the price*

²A complete proof is in the proof of Lemma 4.

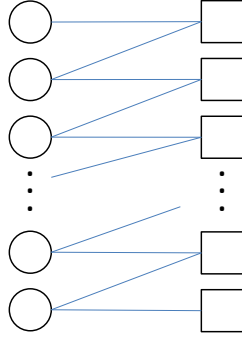


Fig. 2: Structure of bipartite graph at the element-wise maximum market clearing prices

of web slot j in is given by $\sum_{i=j}^{m-1} w_i(c_i - c_{i+1}) + w_m c_m$ for all $j \in M$.

Furthermore, at the maximum market clearing price buyer i prefers web slots i and $i + 1$ for all $i < m$; and buyer m only prefers web slot m if $c_j \neq c_k, w_j \neq w_k$ for all $j \neq k$. The structure of the bipartite graph that is obtained from the prices in Corollary 1 is as shown in Fig. 2 when all elements in vector of c, w are distinct..

We construct this with the market clearing price generated by the VCG mechanism for the sponsored search market.

Corollary 2. *The market clearing price derived by the VCG mechanism is $P_i = \sum_{i=j+1}^{m-1} w_{i+1}(c_i - c_{i+1})$ for all $i \in M$.*

B. Market clearing price return by Algorithm 2

Given knowledge of both the finite-round termination of Algorithm 2 and the set of market clearing prices in a sponsored search market, we would like to determine the market clearing price obtained by the descending price algorithm upon termination. Since the descending price auction is a generalization of the first-price auction, we may conjecture that the algorithm will return the maximum market clearing price vector as given in Corollary 1. The formal statement is in Theorem 2.

Theorem 2. *The proposed descending price algorithm in Algorithm 2 always returns the maximum market clearing price vector.*

Proof: Please see Appendix K in [22]. ■

Prior of proving the theorem, we first try to explain the intuition behind the proof. For ease of explanation we will assume that all the components in vector of c and w are distinct. Given the structure of bipartite graph resulting from the maximum market clearing price vector as depicted in Fig. 2, if Theorem 2 were to be true, the bipartite graph at the termination of Algorithm 2 must have the same structure. This kind of structure indicates buyer i will prefer both slot i and $i + 1$ if $i \neq m$, and buyer m will only prefer slot m . Since the structure bounds the difference of P_i and P_{i+1} , proving the final bipartite structure is equivalent to prove the

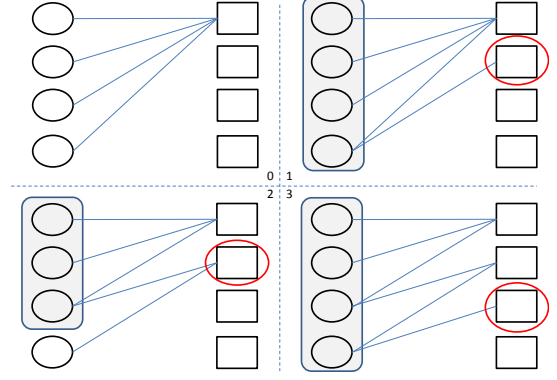


Fig. 3: Forward trace in 4 by 4 sponsored search market

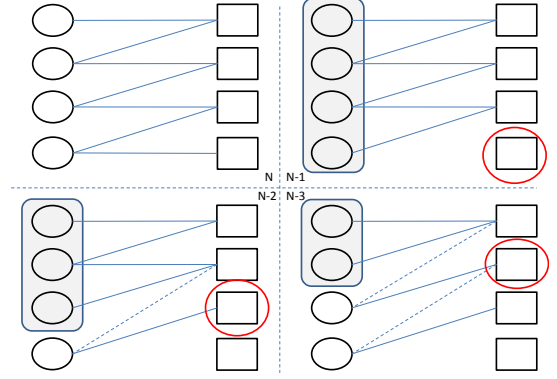


Fig. 4: Backward trace in 4 by 4 sponsored search market

statement below

$$\begin{aligned} P_i - P_{i+1} &= w_i(c_i - c_{i+1}) \text{ for } i < m, \\ P_m &= w_m c_m \end{aligned} \quad (2)$$

As this statement is hard to prove directly, to garner a better understanding of the running of the algorithm, next we present the complete execution of it on an instance with $M = 4$ in Figures 3 and 4.

From Figures 3 and 4, the first observation is the most skewed set has size J contains exactly the web slots from index 1 to J . In each iteration of our algorithm, the price reduction changes the difference of price between any good in the most skewed set and any good not in the most skewed set. If this observation is generally true, we can guarantee that when we change the difference between P_i and P_j , suppose $i < j$ without loss of generality. The price difference between P_i and P_k remains the same for all $k < i$. Incidentally, since the price remains unchanged for every good not in the most skewed set, the The price difference between P_j and P_k remains the same for all $k > j$.

The second observation, the set of neighbors of the most skewed set has size I contains exactly the buyers from index 1 to I , is similar to the first observation. If this observation is true, it alludes to a fact that buyer K will be the last one to be added in the most skewed set with size K .

The last observation is that for all the most skewed set with size J and size of its neighbor I , the buyer $I + 1$ will

be added. Besides, the buyer $I + 1$ originally prefers good $J+1$. If it is true in general, we will know the price difference between P_J and P_{J+1} after the price-reducing procedure will be $w_I(c_J - c_{J+1})$.

These observation 1,2,3 correspond to Lemma 5, 6, 7, respectively.

Lemma 5. *If the most skewed reverse constricted set of web slots S^* has size K , it will contain slots with index from 1 to K .*

Proof: Please see Appendix F in [22]. ■

Lemma 6. *If buyer i is a neighbor of the most skewed reverse constricted set, buyer j must also be a neighbor of the most skewed reverse constricted set for all j with $w_j \geq w_i$.*

Proof: Please see Appendix H in [22]. ■

Lemma 7. *If the most skewed reverse constricted set has size $|S^*| = J$ and $|N_k(S^*)| = I$ at $t = k$, the price reduction in this round should be $w_{I+1}(c_J - c_{J+1}) - P_J^k + P_{J+1}^k$ with $w_{m+1} = c_{m+1} = P_{m+1}^k = 0$ by definition.*

Proof: Please see Appendix I in [22]. ■

If all three observations are true, we will be able to determine the relationship between P_j to P_{j+1} in the final bipartite graph. Since the second observation implies the buyer J will be the last buyer to be added into the most skewed set has size J and observation 3 is general for any I, J . The last iteration to make the most skewed set to be a non-reverse constricted set will anchor the price difference between P_J and P_{J+1} at $w_J(c_J - c_{J+1})$. Then, the first observation tells us the price difference between P_J and P_{J+1} will not be influenced by any price change containing P_k for any $k \neq J$, the price difference between P_J and P_{J+1} in the final bipartite graph is $w_J(c_J - c_{J+1})$ for all $J < m$. As the difference between P_J and P_{J+1} for all $J < m$ matches the statements in Theorem 2, Theorem 2 is affirmed after checking the price $P_m = w_m c_m$ when these three observations are generally true.

At the end of this section, we want to address an additional property in running of the algorithm.

Lemma 8. *Given the skewed-aid descending price algorithm, every bipartite graph is connected in sponsored search markets.*

Proof: Please see Appendix L in [22]. ■

V. DISCUSSIONS

A. Choice of reverse constricted set

Suppose we do not choose the most skewed set in our algorithm, then it follows that we may obtain a different market clearing price vector. However, a natural question to ask is the following: if we run the algorithm two times and choose different reverse constricted sets at some iterations such that the bipartite graph produced in every round of these two executions are the same, will we get the same market clearing price vector?

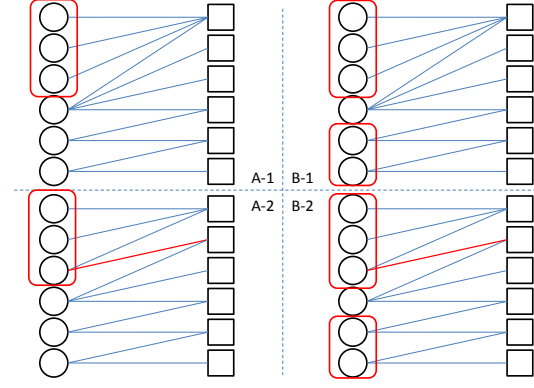


Fig. 5: Counterexample of same bipartite graph but different set market clearing prices

Unfortunately, the choice of the same initial price vector, the same price reduction rule, and the emergence of the same bipartite graph in every round are still not enough to guarantee the same returned market clearing prices. A counterexample is provided in Fig. 5.

In Fig. 5, the bipartite graphs A-1, B-1 have the same lined-up buyer list graph. Though the chosen reverse constricted sets in A,B are different, they add the same buyer to the constricted graph. Therefore, the lined-up buyer list graphs in A-2, B-2 are still the same. However, the updated price vector of A-2, B-2 must be different. If A,B choose the same reverse constricted sets in all the following rounds, the returned sets of market clearing prices of A, B must be different. This example shows that just the bipartite graphs in every round cannot uniquely determine the market clearing price vector obtained at the termination of the algorithm.

B. Comparison with VCG mechanism for general valuations

For a general valuation matrix, the proposed algorithm and the VCG mechanism may not determine the same final bipartite graph. However, when $|B| = |M| = 2$ and $v_{1,1} + v_{2,2} = v_{1,2} + v_{2,1}$, we show that for every valuation matrix, both the VCG mechanism and our algorithm produce the same market clearing matching and same bipartite graph upon termination. First, we study the case of $v_{1,1} + v_{2,2} = v_{1,2} + v_{2,1}$ holds.

Without loss of generality, we can assume $v_{1,1}$ is the largest valuation in V . Then, there are two possible inequalities:

$$v_{1,1} \geq v_{1,2} \geq v_{2,1} \geq v_{2,2} \quad (3)$$

$$v_{1,1} \geq v_{2,1} \geq v_{1,2} \geq v_{2,2} \quad (4)$$

The VCG mechanism has the market clearing price $P_1 = v_{2,1} - v_{2,2}$ and $P_2 = 0$, each buyer is indifferent to each good. The bipartite graph has four edges connecting each buyer and good, and possesses two perfect matchings.

Then, the proposed algorithm has initial price $P_1 = v_{1,1}$ and $P_2 = v_{1,2}$. Since $v_{1,1} + v_{2,2} = v_{1,2} + v_{2,1}$, $v_{1,1} - v_{2,1} = v_{1,2} - v_{2,2}$ implies the proposed algorithm will simultaneously add two edges to connect buyer 2 to the two

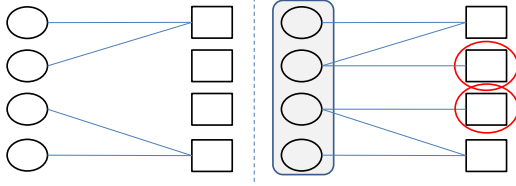


Fig. 6: An example that both initial and final bipartite graphs are disconnected

goods. Therefore, the bipartite graph has four edges the same as the result in VCG mechanism. The only difference is the market clearing price set in our algorithm $\mathbf{P} = [v_{2,1}, v_{2,2}]$ is the maximum market clearing price, but the VCG mechanism returns the lowest market clearing price.

Now, given the social welfare maximization property of market clearing prices and the uniqueness of the social welfare maximizing match when $v_{1,1} + v_{2,2} \neq v_{1,2} + v_{2,1}$, the market clearing matching of VCG mechanism and our algorithm are the same. However, we want to provide a counterexample to show that VCG mechanism and our algorithm may not have the same bipartite graph upon termination when $v_{1,1} + v_{2,2} \neq v_{1,2} + v_{2,1}$.

Assume the inequality in (3) holds and $v_{1,1} + v_{2,2} > v_{1,2} + v_{2,1}$. The VCG mechanism has the market clearing price vector $\mathbf{P} = [v_{2,1} - v_{2,2}, 0]$ and edges $(B_1, M_1), (B_2, M_1), (B_2, M_2)$. However, the algorithm terminates at price vector $\mathbf{P} = [v_{1,1} - v_{1,2} + v_{2,2}, v_{2,2}]$; and the bipartite graph has edges $(B_1, M_1), (B_1, M_2), (B_2, M_2)$. This example shows even both the VCG mechanism and our algorithm have the same matching, the final bipartite graph may not be the same when the condition $v_{1,1} + v_{2,2} = v_{1,2} + v_{2,1}$ does not hold.

C. Failure of Connectivity in Matching Markets

In Lemma 8, we addressed the connectivity property in sponsored search market within the running of our algorithm. Unfortunately, the connectivity does not always hold in general matching markets. Here we demonstrate a 4-by-4 counter example below.

To make this case as simple as possible, we just consider a realized valuation matrix

$$V = \begin{bmatrix} 5 & 4 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 4 & 5 \end{bmatrix}.$$

Initially, the first two goods have lined-up buyer 1, and the last two goods have lined-up buyer 4. In our algorithm, the most skewed set contains all goods, and price is reduced by one in the first round. After that, the market clearing price vector $\mathbf{P} = [4, 3, 3, 4]$ and the bipartite graphs of initial and final steps are in Fig. 6.

Fig. 6 demonstrates that both the initial and final bipartite graphs are disconnected in our algorithm as it has two components. Therefore, preserving the connectivity property in the running of our algorithm only holds in sponsored search markets.

VI. FUTURE WORK

There are still some open problems related to this work. First, since the algorithm returns the set of element-wise maximum market clearing price in sponsored search market, the property/structure of the returned set of market clearing price in general valuation matrix is still an open problem. Allied to this is the question of whether there exists an auction procedure that will produce the maximum market clear price for general valuations. Second, in Section V-B, we find out a special case that both the VCG mechanism and our algorithm have the same bipartite graph. Determining general conditions under which the proposed algorithm will have the same bipartite graph as the VCG mechanism is also an interesting problem. Third, since the proposed algorithm returns the set of element-wise maximum market clearing price in sponsored search market, we would like to understand whether the constricted set in the ascending price algorithm from [6] can be chosen in an intelligent manner so as to produce the same prices as the VCG mechanism. We know that with strategic concerns, the buyers will not bid truthfully in Dutch auction. Hence, given the distribution of each buyer's private weight, we would like to determine the (symmetric) equilibrium bidding strategy in a Bayes-Nash equilibrium given the proposed algorithm? Finally, comparing the bidding strategy in our mechanism with the VCG mechanism, ladder auction proposed in [24], and some other mechanisms in sponsored search market are also future directions to explore.

VII. CONCLUSION

In this paper, we proposed a descending price algorithm in search of sets of the maximum element of the set of market clearing prices. The algorithm is shown to terminate in finite rounds for any non-negative valuation matrix. Furthermore, we defined the skewness of a set, and proved that choosing the most skewed set in each round of price reduction will return the set of element-wise maximum market clearing price in sponsored search markets. Last, we discussed some special cases and presented several possible avenues for future research.

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A. Proof of Lemma 1

First, we want to prove that reducing that amount will increase the set of preferred buyers.

Consider a set of goods \mathbf{S} and the set of their preferred buyers $N(\mathbf{S})$ under a specific price vector P . If \mathbf{S} is a constricted set, consider another price vector P' , let $p'_i = P_i \forall i \notin \mathbf{S}$ and $p'_i = P_i - \min_{j \in M \setminus N(\mathbf{S}), l \in \mathbf{S}} \{\max_{k \in B \setminus \mathbf{S}} (v_{j,k} - P_k) - (v_{j,l} - P_l)\} \forall i \in \mathbf{S}$.

For each $i \in \mathbf{S}$, there must exist a $i = k, j$ satisfying $v_{j,i} - P'_i = \max_k (v_{j,k} - P_k)$. For this $j \in M \setminus N(\mathbf{S})$, $j \in N'(\mathbf{S})$.

For those $n \in N(\mathbf{S})$, since $\max_{n \in \mathbf{S}} (v_{j,n} - p'_n) > \max_{n \in \mathbf{S}} (v_{j,n} - P_n) \geq \max_{k \in B \setminus \mathbf{S}} (v_{j,k} - P_k)$. Therefore, $n \in N'(\mathbf{S})$. So, $N(\mathbf{S})$ is a strictly subset of $N'(\mathbf{S})$, i.e. $N(\mathbf{S}) \subset N'(\mathbf{S})$.

Then, the second step is to prove the reduced price is smaller than this amount, no new buyers will be added to the constricted set.

Let $c = \min_{j \in M \setminus N(\mathbf{S}), l \in \mathbf{S}} \{\max_{k \in B \setminus \mathbf{S}} (v_{j,k} - P_k) - (v_{j,l} - P_l)\}$.

Let $P_j - p'_j < c$, $v_{i,j} - P'_j < v_{i,j} - P_j + c \leq \max_k (v_{i,k} - P_k)$. Therefore, for all $i \in M \setminus N(\mathbf{S})$, $i \notin N'(\mathbf{S})$. In other words, no buyers will be added to the constricted set. With these two steps, we know that if we pick reverse constricted set \mathbf{S} , $\min_{i \in B \setminus N(\mathbf{S}), l \in \mathbf{S}} \{\max_{k \in M \setminus \mathbf{S}} (v_{i,k} - P_k) - (v_{i,l} - P_l)\}$ will be the minimum reduction price.

B. Proof of Lemma 2

By definition, the reverse constricted set S has a property $|S| > |N(S)|$. Since $|S|$, $|N(S)|$ are integers, the skewness of a reverse constricted set

$$f(S) = |S| - |N(S)| + \frac{1}{|S|} \geq 1 + \frac{1}{|S|} > 1. \quad (5)$$

Then, for any non-reverse constricted set S' , $|S'| \leq |N(S')|$. The skewness of S' is

$$f(S') = |S'| - |N(S')| + \frac{1}{|S'|} \leq 0 + \frac{1}{|S'|} \leq 1. \quad (6)$$

With equation (5), (6), the skewness of a reverse constricted set is always greater than any non-reverse constricted sets. Therefore, if a bipartite graph exists a reverse constricted set, the most skewed set is a reverse constricted set.

C. Proof of Lemma 3

We want to prove by contradiction. Suppose there exists two set S_1 , S_2 and both of them are the most skewed, $S_1 = S_2 = \max_{S \in M, S \neq \emptyset} f(S)$.

If S_1, S_2 are disjoint,

$$f(S_1 \cup S_2) \quad (7)$$

$$= |S_1 \cup S_2| - |N(S_1 \cup S_2)| + \frac{1}{|S_1 \cup S_2|} \quad (8)$$

$$= |S_1| + |S_2| - |N(S_1) \cup N(S_2)| + \frac{1}{|S_1 \cup S_2|} \quad (9)$$

$$\geq |S_1| + |S_2| - |N(S_1)| - |N(S_2)| + \frac{1}{|S_1 \cup S_2|} \quad (10)$$

$$= f(S_1) + |S_2| - |N(S_2)| - \frac{1}{|S_1|} + \frac{1}{|S_1 \cup S_2|} \quad (11)$$

$$\geq f(S_1) + 1 - \frac{1}{|S_1|} + \frac{1}{|S_1 \cup S_2|} > f(S_1) \quad (12)$$

From (11) to (12) is true because S_2 is a reverse constricted set. (12) contradicts our assumption that S_1, S_2 are the most skewed set. Therefore, S_1, S_2 are not disjoint. Then, since both S_1, S_2 are the most skewed set, $f(S_1) \geq f(S_1 \cup S_2)$ and $f(S_2) \geq f(S_1 \cap S_2)$.

$$f(S_1) + f(S_2) - f(S_1 \cup S_2) - f(S_1 \cap S_2) \quad (13)$$

$$= |S_1| + |S_2| - |S_1 \cup S_2| - |S_1 \cap S_2| + |N(S_1 \cup S_2)| + |N(S_1 \cap S_2)| - |N(S_1)| - |N(S_2)| + \frac{1}{|S_1|} + \frac{1}{|S_2|} - \frac{1}{|S_1 \cup S_2|} - \frac{1}{|S_1 \cap S_2|} \quad (14)$$

The first four terms $|S_1| + |S_2| - |S_1 \cup S_2| - |S_1 \cap S_2| = 0$. Using the similar argument, $|N(S_1)| + |N(S_2)| = |N(S_1) \cap N(S_2)| + |N(S_1) \cup N(S_2)|$.

$$|N(S_1 \cup S_2)| = |N(S_1) \cup N(S_2)| \quad (15)$$

$$|N(S_1 \cap S_2)| \leq |N(S_1) \cap N(S_2)| \quad (16)$$

(16) is true because there may exist some elements in $S_1 \setminus S_2$ and $S_2 \setminus S_1$ but have common neighbors. Thus, the second four terms are smaller than or equal to 0.

To check the last four terms, we have to argue that $|S_1| = |S_2|$ first. If $|S_1| \neq |S_2|$, the difference between $\frac{1}{|S_1|}$ and $\frac{1}{|S_2|}$ is a fraction. Since $S_1, S_2, N(S_1), N(S_2)$ are integers, $|S_1| \neq |S_2|$ makes $f(S_1) \neq f(S_2)$. So, $|S_1| = |S_2|$.

Then, let $|S_1| = a, |S_1 \cup S_2| - |S_1| = b$, the last four terms are

$$\frac{1}{|S_1|} + \frac{1}{|S_2|} - \frac{1}{|S_1 \cup S_2|} - \frac{1}{|S_1 \cap S_2|} \quad (17)$$

$$= \frac{1}{a} + \frac{1}{a} - \frac{1}{a+b} - \frac{1}{a-b} \quad (18)$$

$$= \frac{a}{a^2} + \frac{a}{a^2} - \frac{a-b}{a^2-b^2} - \frac{a+b}{a^2-b^2} \quad (19)$$

$$= \frac{2a}{a^2} - \frac{2a}{a^2-b^2} < 0 \quad (20)$$

Therefore, the last four terms are negative. To conclude, the first four terms are 0, the second four terms are smaller than or equal to 0, and the last four terms are negative make

$$f(S_1) + f(S_2) - f(S_1 \cup S_2) - f(S_1 \cap S_2) < 0. \quad (21)$$

It implies at least one of $f(S_1 \cup S_2), f(S_1 \cap S_2)$ greater than $f(S_1) = f(S_2)$, which leads to a contradiction that S_1 and S_2 are the most skewed set. Therefore, the most skewed set is unique.

D. Proof of Theorem 1

The relationship between S_k^* and S_{k+1}^* has four possible cases.

- 1) $S_k^* = S_{k+1}^*$
- 2) $S_k^* \subset S_{k+1}^*$
- 3) $S_k^* \supset S_{k+1}^*$
- 4) $S_k^* \not\subseteq S_{k+1}^*$ and $S_k^* \not\supseteq S_{k+1}^*$

In case 1, $G_k - G_{k+1} = f_k(S_k^*) - f_{k+1}(S_{k+1}^*) = f_k(S_k^*) - f_{k+1}(S_k^*) \geq 1$.

In case 2, define $S' = S_{k+1}^* \setminus S_k^*$, $N_k(S)$ be the neighbor of S based on the bipartite graph at round k .

By definition, $f_k(S_k^*) - f_k(S_{k+1}^*) > 0$.

$$f_k(S_k^*) - f_k(S_{k+1}^*) = |S_k^*| - |N_k(S_k^*)| - |S_{k+1}^*| + |N_k(S_{k+1}^*)| + \frac{1}{|S_k^*|} - \frac{1}{|S_{k+1}^*|} > 0. \text{ Since } S_k^* \subset S_{k+1}^*, 1 > \frac{1}{|S_k^*|} - \frac{1}{|S_{k+1}^*|} > 0.$$

$$\text{Therefore, } |S_k^*| - |N_k(S_k^*)| \geq |S_{k+1}^*| - |N_k(S_{k+1}^*)|.$$

$$G_k - G_{k+1} \quad (22)$$

$$= f_k(S_k^*) - f_{k+1}(S_{k+1}^*) \quad (23)$$

$$= |S_k^*| - |N_k(S_k^*)| - |S_{k+1}^*| + |N_{k+1}(S_{k+1}^*)| + \frac{1}{|S_k^*|} - \frac{1}{|S_{k+1}^*|} \quad (24)$$

$$\geq |S_{k+1}^*| - |N_k(S_{k+1}^*)| - |S_{k+1}^*| + |N_{k+1}(S_{k+1}^*)| + \frac{1}{|S_k^*|} - \frac{1}{|S_{k+1}^*|} \quad (25)$$

$$= |N_{k+1}(S_k^* \cup S')| - |N_k(S_{k+1}^*)| + \frac{|S_{k+1}^*| - |S_k^*|}{|S_{k+1}^*||S_{k+1}^*|} \geq |N_{k+1}(S_k^* \cup S')| - |N_k(S_{k+1}^*)| + \frac{1}{M(M-1)} \quad (26)$$

$$= |N_{k+1}(S_k^*)| + |N_{k+1}(S') \cap N_{k+1}(S_k^*)^c| - |N_k(S_{k+1}^*)| + \frac{1}{M(M-1)} \quad (27)$$

$$= |N_k(S_k^*)| + |N_{k+1}(S_k) \setminus N_k(S_k^*)| - |N_k(S_{k+1}^*)| + |N_{k+1}(S') \cap N_{k+1}(S_k^*)^c| + \frac{1}{M(M-1)} \quad (28)$$

Before going to further steps, we have to briefly explain the logic behind the above equations.

(24) to (25) is true because $N_k(S_k^*) \subseteq N_k(S_{k+1}^*)$.

(26) to (27) is to expand $|N_{k+1}(S_k^* \cup S')|$ to $|N_{k+1}(S_k^*)| + |N_{k+1}(S') \cap N_{k+1}(S_k^*)^c|$.

(27) to (28) is to expand $|N_{k+1}(S_k^*)|$ to $|N_k(S_k^*)| + |N_{k+1}(S_k) \setminus N_k(S_k^*)|$.

Since $|N_{k+1}(S') \cap N_{k+1}(S_k^*)^c| = |N_{k+1}(S') \cap N_k(S_k^*)^c| - |N_{k+1}(S') \cap (N_{k+1}(S_k) \setminus N_k(S_k^*))|$, and $|N_{k+1}(S_k) \setminus N_k(S_k^*)| \geq |N_{k+1}(S') \cap (N_{k+1}(S_k) \setminus N_k(S_k^*))|$

$$\begin{aligned}
& |N_k(S_k^*)| + |N_{k+1}(S_k) \setminus N_k(S_k^*)| \\
& + |N_{k+1}(S') \cap N_{k+1}(S_k^*)^c| - |N_k(S_{k+1}^*)| \quad (29) \\
\geq & |N_k(S_k^*)| + |N_{k+1}(S') \cap N_k(S_k^*)^c| \\
& - |N_k(S_{k+1}^*)| \quad (30) \\
= & |N_k(S_k^*)| + |N_{k+1}(S') \cap N_k(S_k^*)^c| \\
& - |N_k(S_k^*)| - |N_k(S') \cap N_k(S_k^*)^c| \quad (31) \\
= & |N_{k+1}(S') \cap N_k(S_k^*)^c| - |N_k(S') \cap N_k(S_k^*)^c| \quad (32)
\end{aligned}$$

Then, we want to claim that

$$|N_{k+1}(S') \cap N_k(S_k^*)^c| = |N_k(S') \cap N_k(S_k^*)^c|.$$

Since S' is not in S_k^* , the neighbor of S' not in the neighbor of S_k^* remains the same from k to $k+1$ round. Therefore, the equation (28) is greater than or equal to $\frac{1}{M(M-1)}$.

In case 3, since every elements in S_k^* belongs to S_{k+1}^* , $f_k(S_{k+1}^*) \geq f_{k+1}(S_{k+1}^*)$.

Then, by definition, $f_k(S_k^*) > f_k(S_{k+1}^*)$. Since $\frac{1}{|S_k^*|} < \frac{1}{|S_{k+1}^*|}$, $f_k(S_k^*) - f_k(S_{k+1}^*) > 0$ implies $|S_k^*| - |S_{k+1}^*| + |N_k(S_k^*)| - |N_k(S_{k+1}^*)| \geq 1$.

Therefore, $f_k(S_k^*) - f_k(S_{k+1}^*) \geq \frac{1}{2}$.

$$\begin{aligned}
G_k - G_{k+1} &= f_k(S_k^*) - f_{k+1}(S_{k+1}^*) \\
&\geq f_k(S_k^*) - f_k(S_{k+1}^*) \geq \frac{1}{2}. \quad (33)
\end{aligned}$$

In case 4, define $S' = S_{k+1}^* \setminus S_k^*$, $S'' = S_k^* \setminus S_{k+1}^*$, and $T = S_k^* \cap S_{k+1}^*$.

$$G_k - G_{k+1} \quad (34)$$

$$= f_k(S_k^*) - f_{k+1}(S_{k+1}^*) \quad (35)$$

$$\begin{aligned}
&= |T| + |S''| - |N_k(T)| - |N_k(S'') \setminus N_k(T)| + \frac{1}{|S_k^*|} \\
&\quad - |T| - |S'| + |N_{k+1}(S_{k+1}^*)| - \frac{1}{|S_{k+1}^*|} \quad (36)
\end{aligned}$$

$$\begin{aligned}
&= |S''| - |N_k(S'') \setminus N_k(T)| + \frac{1}{|S_k^*|} - \frac{1}{|S_{k+1}^*|} \\
&\quad + |N_{k+1}(S_{k+1}^*)| - |N_k(T)| - |S'| \quad (37)
\end{aligned}$$

$$\begin{aligned}
&\geq 1 + \frac{1}{|S_k^*|} - \frac{1}{|S_{k+1}^*|} + |N_{k+1}(S_{k+1}^*)| \\
&\quad - |N_k(T)| - |S'| \quad (38)
\end{aligned}$$

$$\begin{aligned}
&\geq 1 + \frac{1}{M} - 1 + |N_{k+1}(T)| - |N_k(T)| \\
&\quad + |N_{k+1}(S') \setminus N_{k+1}(T)| - |S'| \quad (39)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{M} + |N_{k+1}(T)| - |N_k(T)| \\
&\quad + |N_{k+1}(S') \setminus N_{k+1}(T)| - |S'| \quad (40)
\end{aligned}$$

$$\geq \frac{1}{M} + |N_{k+1}(S') \setminus N_k(T)| - |S'| \quad (41)$$

$$\geq \frac{1}{M} \quad (42)$$

Most of the equations in case 4 are straight-forward except from (40) to (41).

(40) to (41) is true because of the following inequalities:

$$\begin{aligned}
& |N_{k+1}(S') \setminus N_{k+1}(T)| + |N_{k+1}(T)| - |N_k(T)| \\
&= |N_{k+1}(S') \setminus N_k(T)| + |N_{k+1}(T)| - |N_k(T)| \\
&\quad - |\{N_{k+1}(S') \cap N_{k+1}(T)\} \setminus N_k(T)| \quad (43)
\end{aligned}$$

$$\begin{aligned}
&\geq |N_{k+1}(S') \setminus N_k(T)| - |N_{k+1}(T) \setminus N_k(T)| \\
&\quad + |N_{k+1}(T)| - |N_k(T)| \quad (44)
\end{aligned}$$

$$= |N_{k+1}(S') \setminus N_k(T)| \quad (45)$$

Combine these four cases, we know that

$$G_k - G_{k+1} \geq \frac{1}{M(M-1)} > 0. \quad (46)$$

Therefore, we can conclude that the proposed algorithm terminates in a finite round.

E. Proof of Lemma 4

In a sponsored search market, assigning buyer i the i^{th} web slot is always a market clearing matching. With this property, we first show that any set of price $P = \{P_1, P_2, \dots, P_M\} \notin \mathbf{Q}$, it is not a set of market clearing price.

First, $P \notin \mathbf{Q}$ implies there exists some $i \in 1, 2, \dots, m-1$ satisfying at least one of the inequalities:

$$P_i - P_{i+1} > w_i c_i - w_i c_{i+1} \quad (47)$$

$$P_i - P_{i+1} < w_{i+1} c_i - w_{i+1} c_{i+1}. \quad (48)$$

If any pair of $(i, i+1)$ satisfies (47), the buyer i will prefer web slot $i+1$ rather than the slot i , which is allocated in perfect matching. Therefore, the set P can not form a set of market clearing prices. If any pair of $(i, i+1)$ satisfies (48), the buyer $i+1$ will prefer web slot i rather than the slot $i+1$, which is allocated in perfect matching. Similarly, the set P can not form a set of market clearing prices.

Then, we have to show that every $P \in \mathbf{Q}$, it can form a set of market clearing prices.

At perfect matching, the buyer i gets the i^{th} web slot. for any pair (i, j) , $i < j$, the following equation must holds:

$$P_i - P_j \leq w_i c_i - w_j c_j. \quad (49)$$

It is straight forward to verify. for any (i, j) , $P_i - P_j > w_i c_i - w_j c_j$, the buyer i will prefer web slot j rather than slot i . It does not belong to the market clearing allocation. Furthermore, the following equation also have to hold:

$$P_i - P_j \geq w_j c_i - w_j c_j. \quad (50)$$

Similarly, if $P_i - P_j < w_j c_i - w_j c_j$, the buyer j will prefer slot i rather than j . Combing (49), (50), for each pair (i, j) , there must be boundary of $P_i - P_j$ that

$$w_j(c_i - c_j) \leq P_i - P_j \leq w_i(c_i - c_j). \quad (51)$$

Now, if $j - i > 1$; there exists a k , $i < k < j$. Using the inequality in (51), we can derive the following inequality:

$$\begin{aligned}
w_k(c_i - c_k) + w_j(c_k - c_j) &\leq (P_i - P_k) + (P_k - P_j) \\
&\leq w_i(c_i - c_k) + w_k(c_k - c_j). \quad (52)
\end{aligned}$$

Then, add $(w_i c_j - w_i c_j) = 0$ to the right part of (52)

$$w_i(c_i - c_k) + w_k(c_k - c_j) + (w_i c_j - w_i c_j) \\ = w_i(c_i - c_j) - (w_i - w_k)(c_k - c_j) \leq w_i(c_i - c_j) \quad (53)$$

Similarly, add $(w_j c_i - w_j c_i) = 0$ to the left part of (53)

$$w_k(c_i - c_k) + w_j(c_k - c_j) + (w_j c_i - w_j c_i) \\ = w_j(c_i - c_j) + (w_k - w_j)(c_k - c_j) \geq w_i(c_i - c_j) \quad (54)$$

With (53),(54), we can summarize that for all pair (i,j), $|i - j| > 1$, the constraint of pair (i,j) is a subset of constraints (i,k) and (k,j), $i < k < j$. Therefore, we can remove all constraints with $|i - j| > 1$. Now, the remaining constraints of the market clearing price P is for all $i \in 1, 2, \dots, m$

$$w_{i+1}(c_i - c_{i+1}) \leq P_i - P_{i+1} \leq w_i(c_i - c_{i+1}). \quad (55)$$

Then, there's an extra constraint that the dummy parameters $P_{m+1} = w_{m+1} = c_{m+1} = 0$ in the set of market clearing price. Now, it is equivalent to the set \mathbf{Q} . Therefore, every $P \in \mathbf{Q}$ can form a set of market clearing price. To summarize, for all $P \notin \mathbf{Q}$ cannot be a set of market clearing price and every $P \in \mathbf{Q}$ is a set of market clearing price, the proof is complete here.

F. Proof of Lemma 5

In order to prove Lemma 5, we need the result of Lemma 9. It is the non-separable property under our algorithm in sponsored search markets. We suggest the reader to go through the statement in Lemma 5 before reading the proof of this Lemma.

Suppose the set S^* with size K contains at least one slots with index greater than K , there must exists a $l < K$, slot L_l not in S^* .

By definition, we know that $|N(L_l) \setminus N(S)| \geq 1$. For any $i \in N(L_l) \setminus N(S)$,

$$w_i c_l - P_l > w_i c_j - P_j, \quad \forall j \in S. \quad (56)$$

Therefore, $\exists j > K, j \in S$ such that $w_m c_j - P_j \geq w_m c_l - P_l$ for some $m \in N(S)$

Now, we have

$$w_i(c_l - c_j) > P_l - P_j \geq w_m(c_l - c_j) \quad (57)$$

Since $P_l \geq P_j$ for all $l < j$, all the neighbor of L_j has private weight $w_m < w_i$.

Similarly, if $l > 1$, $\exists q < l, q \in S$ such that $w_n c_q - P_q \geq w_n c_l - P_l$ for some $n \in N(S)$ Hence, we can get a similar inequality that

$$w_n(c_q - c_l) \geq P_q - P_l > w_i(c_q - c_l) \quad (58)$$

We know that $P_q \geq P_l$ for all $l > q$, all the neighbor of L_q has private weight $w_n > w_i$. If we have this kind of reverse constricted set, it will not be the most skewed one by Lemma 9.

Then, we have to prove $l = 1$ will not happen.

Initially, slot 1 and buyer 1 are in the most skewed reverse constricted set. Besides, buyer 1 is the only the lined-up buyer of slot 1. Excluding the trivial case that every slots in the

most skewed set has the same weight as slot 1, there exists at least a slot j has $c_j < c_1$ has a common lined-up buyer with slot 1, $w_1 c_1 - P_1 = w_1 c_j - P_j$ when slot 1 is in the most skewed reverse constricted set. If $l = 1$ could first happen at time $t = q$, at least a buyer i with $w_i < w_1$ would be added to the lined-up buyer list of L_1 at $t = r < q$. It implies at $t = r + 1$, there exists a $n \in N_{q+1}(B_i) \setminus N_q(S_q^*)$ such that $w_i c_n - P_n = w_i c_1 - P_1 \geq w_i c_j - P_j$. It implies that $w_i(c_1 - c_j) \geq P_1 - P_j$. However, the fact that slot 1 and j has common buyer 1 guarantees $w_1 c_1 - P_1 = w_1 c_j - P_j$. There is a contradiction among $w_i(c_1 - c_j) \geq P_1 - P_j$, $w_1(c_1 - c_j) = P_1 - P_j$, and $w_i < w_1$. Therefore, $l = 1$ will not happen.

G. Lemma 9 and its proof

The Lemma 9 states the non-separable property in sponsored search markets under our algorithm. The formal statement is appended and proved below:

Lemma 9. *In sponsored search markets, the most skewed set cannot be separated into two (or more) sets with disjoint neighbors under the reducing price policy of our algorithm.*

Proof: At $t = 0$, every slot has the same lined-up buyer list. We have shown that the algorithm terminates in a finite round. Suppose there exist a finite integer T , $t = T$ is the first time that the most skewed set S_T^* can be separated into two sets S_1, S_2 with disjoint neighbors. (At this time, both S_1, S_2 are reverse constricted set.)

We know a general rule that the price reduction of every web slot in a set S will remove their neighbors from the lined-up-buyer list of any slots not in this set. That is, $N(S^C) \cap N(S) = \emptyset$.

Since the most skewed set cannot be separated into two sets initially, there exists a $0 < t_1 < T$. At t_1 , the most skewed set $S_{t_1}^*$ contains one of S_1, S_2 but disjoint with another. $S_1 \subseteq S_{t_1}^*, S_{t_1}^* \cap S_2 = \emptyset$ or $S_2 \subseteq S_{t_1}^*, S_{t_1}^* \cap S_1 = \emptyset$.

Suppose $S_1 \subseteq S_{t_1}^*$, any subset $S' \subseteq S_2$ has the property that $|S'| \leq |N_{t_1}(S') \setminus N_{t_1}(S_{t_1}^*)|$ at $t = t_1$. If it was not, $S_{t_1}^* \cup S'$ would have a higher skewness than $S_{t_1}^*$.

Since any subset of S_2 is not a reverse constricted set at $t = t_1$, S_2 is not a reverse constricted set at $t \geq t_1$. (Suppose for some t_k , any subset of S_2 is not a reverse constricted set. Assume $S_3 \subseteq S_2, S_3 \in S_{t_k}^*$ and $S_2 \setminus S_3 \notin S_{t_k}^*$, at $t = t_k + 1$, any subset of $S_2 \setminus S_3$ is not a reverse constricted set since $\{S_2 \setminus S_3\} \notin S_{t_k}^*$, and any subset of S_3 is still not a reverse constricted set since no edged of S_3 has been removed. Therefore, we can conclude that if any subset of S_2 is not a reverse constricted set at $t = t_1$, any subset of S_2 will not a reverse constricted set at $t > t_1$ by mathematical induction.)

Since S_2 is not a reverse constricted set at $t \geq t_1$ leading to a contradiction that S_2 is a reverse constricted set at $t = T$. The most skewed set is not separable. ■

H. Proof of Lemma 6

If $i \in N(S)$, $\exists k \in S$ such that $w_i c_k - P_k \geq w_i c_l - P_l, \forall l \notin S$.

It implies that $w_i(c_k - c_l) \geq P_k - P_l, \quad \forall l \notin S$.

With Lemma 5, the most skewed reverse constricted set contains slots from 1 to its size. It implies that $c_k > c_l$.

Therefore, for all $w_j > w_i, w_j(c_k - c_l) > w_i(c_k - c_l) \geq P_k - P_l \quad \forall l \notin S$. The proof is done here.

I. Proof of Lemma 7

The proof of this lemma will use Lemma 6 and Lemma 10. Lemma 10 is a property in our algorithm that has at least one lined-up buyer.

With Lemma 6, we know that the most skewed reverse constricted set increases the size of its neighbors with an ascending order of buyer's index. It is obvious that the buyer B_{J+1} will be added to the reverse constricted set.

Then, we know that each slots has at least one line-up buyer. Since $L_{I+1} \notin S^*$, B_{J+1} must be in the line-up buyer list of L_{I+1} . If not, since $P_{I+1} - P_O > w_i(c_{I+1} - c_O), \quad \forall O > I + 1, P_{I+1} - P_O > w_j(c_{I+1} - c_O) \forall j > i$. Then slot L_{I+1} will have no line-up buyer. It violates the property of our algorithm. Therefore, the reduced price should at least let the buyer B_{J+1} has no preference between L_I and L_{I+1} .

Finally, we need to show that the added buyer B_{J+1} will prefer L_I . If B_{J+1} will prefer any slots with $c_j > c_I$. It implies that $w_{J+1}c_j - P_j > w_{J+1}c_I - P_I$. Then, for any slot i in the most skewed reverse constricted set, $w_i > w_{J+1}$. Therefore, $w_i(c_j - c_I) > w_{J+1}(c_j - c_I) > P_j - P_I$ implies the lined-up-buyer list of L_I is an empty set, which violates the Lemma 10. Therefore, the newly added buyer B_{J+1} will prefer L_I .

J. Lemma 10 and its proof

Lemma 10. *At each time t , every web slot has at least one lined-up buyer.*

Proof: Initially, every web slots has a lined-up buyer (the buyer 1) at $t = 0$.

Suppose there exists a finite $k, t = k$ is the first time that the slot L_l has no lined-up buyer. It implies that at $t = k - 1, L_l \notin S_{k-1}^*$ but all neighbors of L_l are in the neighbor of the most skewed set at time $k - 1, N_{k-1}(L_l) \subseteq N_{k-1}(S_{k-1}^*)$. Then $f(S_{k-1}^* \cup \{L_l\}) > f(S_{k-1}^*)$ contradicts that S_{k-1}^* is the most skewed one. Therefore, every web slot has at least one lined-up buyer. ■

K. Lemma 11 and its proof

Lemma 11. *For any $w_i > w_j > w_k$ and $c_p > c_q > c_r$, if buyer j is in the lined-up buyer list of good q , the buyer i will not prefer good r and buyer k will not prefer good p .*

Proof: By definition, $w_jc_q - P_q \geq w_jc_r - P_r$. Then, $w_j(c_q - c_r) \geq P_q - P_r$. Since $w_i > w_j, w_i(c_q - c_r) > P_q - P_r$ implies that buyer i will not prefer good r .

Similarly, $w_jc_q - P_q \geq w_jc_p - P_p$. Then, $w_j(c_p - c_q) \leq P_p - P_q$. Since $w_k < w_j, w_k(c_p - c_q) < P_p - P_q$ implies that buyer k will not prefer good p . This lemma indicates that there is no cross links when all the elements in vector c and w are disjoint. ■

L. Proof of Lemma 8

Initially, at $t = 0$, the bipartite graph is connected because every slot has the same lined-up buyer (buyer 1) in our algorithm. Then, we want to prove every bipartite graph is our algorithm is connected by induction.

Suppose at $t = k$, the bipartite graph is connected. Then, consider the bipartite graph at $t = k + 1$.

With the result in Lemma 7, our algorithm always add edges to make the consecutive slots have a common lined-up buyer. Therefore, we can claim that every two consecutive slots are connected at time $t = k$. Then, at $t = k + 1$, the edges connecting the slots in the most skewed reverse constricted set and their neighbors remain unchanged. It is obvious that every slot in the most skewed reverse constricted set is still connected.

Then, with Lemma 11, the common lined-up buyer of consecutive slots not ins the most skewed reverse constricted set must be not in the neighbor of the most skewed reverse constricted set. $L_i, L_{i+1} \notin S_k^*, \{N_k(L_i) \cap N_k(L_{i+1})\} \cap N_k(S_k^*) = \emptyset$. Recall the claim in (32), $|N_{k+1}(S') \cap N_k(S_k^*)^c| = |N_k(S') \cap N_k(S_k^*)^c|$ and the result in Lemma 11, we know that if the slots are not in the most skewed reverse constricted set is connected at $t = k$, they will also connected at $t = k + 1$.

Then, the last part is our algorithm add edges to connect at least a slot in the most skewed reverse constricted set and at least a slot not in the most skewed reverse constricted set. Since all the slots in the most skewed set are connected and all slots not in the most skewed set are also connected at $t = k + 1$, our algorithm add at least an edge to connect two disjoint connected components. Therefore, all the slots in the bipartite graph at $t = k + 1$ are still connected. It completes the proof of the connectivity property in sponsored search markets under the skewed-aid descending price algorithm.